Energy of Capillary Waves on a Canal of Uniform Depth

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Abstract

In this paper, basic concepts of capillary wave motion are presented. Capillary wave on the surface of a canal is discussed. For this wave, when velocity potential is found, its energy is obtained.

Keywords: Capillary wave, Kinetic energy, Potential energy

Introduction

A wave in a fluid is the continuous transfer of a state or form from one part of the fluid medium to another. A disturbance moves through the medium without any transference of medium itself. The dynamics of wave motion is very important. The wave possesses a characteristic property: the energy is propagated to distant points. The transmission of the energy of water waves benefits our lives. Hence, we examine the kinetic energy and potential energy for the capillary waves.

The Condition of Capillary Waves

A **capillary wave** is a wave moving along the phase boundary of a fluid, dominated by the effects of surface tension. It occurs due to any moving disturbance. A combination of gravity and surface tension is here the restoring force. An interface between two fluids performs as if it were in a state of uniform tension, called the **surface tension.** These two fluids do not mix. The surface tension depends on the nature of two fluids and on the temperature.

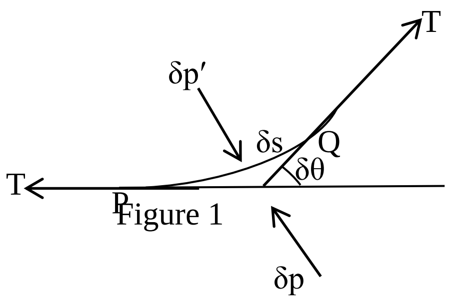

Figure 1

Let PQ = δs be an element of arc of a cross section of a cylindrical surface. The surface is formed by the interface between two fluids and T is its

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surface tension. Let δp and $\delta p'$ denote the pressure below and above the surface respectively and $\delta\theta$ the angle between two tangents at P and Q.

At P, resolving along the normal, we get

$$(\delta p)\delta s\cos\delta\theta - (\delta p')\delta s\cos\delta\theta + T\sin\delta\theta = 0,$$

$$(\delta p)\delta s - (\delta p')\delta s + T\delta\theta = 0, \text{ since } \delta\theta \text{ is small,}$$

$$\delta p - \delta p' + T\frac{\delta\theta}{\delta s} = 0,$$

$$\delta p - \delta p' + \frac{T}{R} = 0, \quad (1)$$

where R is the radius of curvature.

Therefore, there is a discontinuity of pressure at an interface. Let η denotes the vertical displacement of the interface at a point x at any time t. Since the slope is small, the curvature is $\frac{\partial^2\eta}{\partial x^2}$.

That is, $\frac{1}{R} = \frac{\partial^2\eta}{\partial x^2}$. Therefore, (1) becomes

$$\delta p - \delta p' + T\frac{\partial^2\eta}{\partial x^2} = 0. \quad (2)$$

Capillary Waves on a Canal of Uniform Depth

Let the progressive waves be on the surface of canal. It has parallel vertical walls and its uniform depth is h. And let X-axis and Y-axis to be horizontal and vertically upwards respectively. We have the velocity potential

$$\phi = D\cosh m(y+h)\cos mx,$$

where m is the wave number and D the constant .

Let the progressive waves travel with velocity c along the positive direction of X-axis. It is supposed that their shape do not change. By superposing a velocity $-c$ on the whole liquid, we reduce the motion as steady motion. Then the velocity potential is obtained.

$$\phi = cx + D\cosh m(y+h)\cos mx \quad (3)$$

and stream function

$$\psi = cy - D\sinh m(y+h)\sin mx. \quad (4)$$

Let the free surface be $\eta = a\sin mx$. Then (4) will produce this free surface when the streamline is zero. When $\psi = 0$, (4) becomes

$$ca - D\sinh mh = 0, \tag{5}$$

when squares of small quantities are neglected.

Suppose that two fluids be air and liquid and the air pressure above the interface be constant. Then, $\delta p' = 0$ and (2) becomes

$$T\frac{\partial^2\eta}{\partial x^2} + \delta p = 0. \tag{6}$$

But, $\eta = a\sin mx$. (7)

Differentiating (7) with respect to x and substituting in (6), we get

$$\delta p = Tam^2 \sin mx. \tag{8}$$

Now the pressure is given by Bernoulli's equation,

$$\frac{p}{\rho} + gy + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] = \text{constant}. \tag{9}$$

At the free surface $y = a\sin mx$, using (3), (5) and neglecting a^2, (9) becomes

$$\frac{\delta p}{\rho} + ga\sin mx + \frac{1}{2}c^2(1 - 2ma\coth mh\sin mx) = \text{constant}. \tag{10}$$

Substituting (8) in (10), we get

$$\left(\frac{Tam^2}{\rho} + ga - c^2 ma\coth mh\right)\sin mx = \text{constant}.$$

Equating the coefficient of sin mx to be zero, we obtain

$$\frac{Tm^2}{\rho} + g - c^2 m\coth mh = 0,$$

$$c^2 = \left(\frac{Tm}{\rho} + \frac{g}{m}\right)\tanh mh.$$

The Energy of Capillary Waves

Kinetic energy of capillary waves on a canal of uniform depth

The velocity potential of capillary waves on a canal of uniform depth is given by

$$\phi = cx + D\cosh m(y + h)\cos mx.$$

For these waves, the kinetic energy (K.E.) is defined by

$$\text{K.E.} = \frac{1}{2}\rho\int_0^{\lambda}\left(\phi\frac{\partial\phi}{\partial y}\right)_{y=0} dx \text{, where } \lambda \text{ is wave length.}$$

Here, $\left(\phi\frac{\partial\phi}{\partial y}\right)_{y=0}$ $= (\mathrm{Dm}\sinh mh\cos mx)(cx + D\cosh mh\cos mx)$

$$= c^2 amx\cos mx + c^2a^2 m\coth mh\cos^2 mx,$$

since $ca = D\sinh mh$.

Then, $\text{K.E} = \frac{1}{2}\rho\int_0^{\lambda}(c^2 amx\cos mx + c^2a^2 m\coth mh\cos^2 x)dx$,

Since $c^2 = \left(\frac{Tm}{\rho} + \frac{g}{m}\right)\tanh mh$, we obtain

$$\text{K.E.} = \left(\frac{Tm}{\rho} + \frac{g}{m}\right)a^2 m\frac{1}{2}\rho\left[\int_0^{\lambda}\cos mx\tanh mh dx + \int_0^{\lambda}\cos^2 mx\coth mh\tanh mh dx\right].$$

Therefore,

$$\text{K.E.} = \left(\frac{Tm}{\rho} + \frac{g}{m}\right)a^2 m\frac{1}{2}\rho\left[\int_0^{\lambda}\cos mx\tanh mh dx + \int_0^{\lambda}\cos^2 mx dx\right]$$

$$= \left(\frac{Tm}{\rho} + \frac{g}{m}\right)a^2 m\frac{1}{2}\rho\left[\left[\tanh mh\frac{\sin mx}{m}\right]_0^{\lambda} + \frac{1}{2}\left[x + \frac{\sin 2mx}{2m}\right]_0^{\lambda}\right]$$

$$= \left(\frac{Tm}{\rho} + \frac{g}{m}\right)a^2 m\frac{1}{2}\rho\left[\left[\tanh mh\left(\frac{\sin m\lambda}{m} - \sin 0\right)\right] + \frac{1}{2}\left[\lambda + \frac{\sin 2m\lambda}{2m}\right]\right]$$

$$= \left(\frac{Tm}{\rho} + \frac{g}{m}\right)a^2 m\frac{1}{2}\rho\left[\left[\tanh mh\left(\frac{\sin m\frac{2\pi}{m}}{m}\right)\right] + \frac{1}{2}\left[\lambda + \frac{\sin 2m\frac{2\pi}{m}}{2m}\right]\right]$$

$$= \frac{1}{4}(Tm^2 + \rho g)a^2\lambda .$$

Potential energy of capillary waves on a canal of uniform depth

The potential energy of capillary waves on a canal of uniform depth associated with the gravity is given by

$$V = \frac{1}{2}\rho g\int_0^{\lambda}\eta^2 dx$$

$$= \frac{1}{2}\rho g a^2 \int_0^{\lambda} \sin^2 mx dx, \text{ since } \eta = a \sin mx$$

$$= \frac{1}{2}\rho g a^2 \int_0^{\lambda} \frac{1}{2}(1 - \cos 2mx) dx$$

$$= \frac{1}{4}\rho g a^2 \left[x - \frac{\sin 2mx}{2m} \right]_0^{\lambda}$$

$$= \frac{1}{4}\rho g a^2 \left[\lambda - \frac{\sin 2m\lambda}{2m} \right]$$

$$V = \frac{1}{4}\rho g a^2 \left[\lambda - \frac{\sin 2m \frac{2\pi}{m}}{2m} \right]$$

$$= \frac{1}{4}\rho g a^2 \lambda .$$

The potential energy of capillary waves on a canal of uniform depth associated with the surface tension , V_{st} can be expressed as

$$V_{st} = T\int_0^{\lambda} \left[\left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{1/2} - 1 \right] dx$$

$$= \frac{1}{2} T \int_0^{\lambda} \left(\frac{\partial \eta}{\partial x} \right)^2 dx$$

$$= \frac{1}{2} T \int_0^{\lambda} a^2 m^2 \cos^2 mx dx, \text{ since } \eta = a \sin mx$$

$$= \frac{1}{4} T a^2 m^2 \int_0^{\lambda} (1 + \cos 2mx) dx,$$

$$= \frac{1}{4} T a^2 m^2 \left[x - \frac{\sin 2mx}{2m} \right]_0^{\lambda}$$

$$V_{st} = \frac{1}{4} T a^2 m^2 \left[\lambda - \frac{\sin 2m\lambda}{2m} \right]$$

$$= \frac{1}{4} Ta^2 m^2 \lambda.$$

Therefore, the total potential energy is $\frac{1}{4}(Tm^2 + \rho g)a^2\lambda.$

So, total energy per wavelength $= \frac{1}{2}(Tm^2 + \rho g)a^2\lambda$.

Conclusion

It is concluded that the total potential energy of capillary waves is the energy associated with the gravity and surface tension. It is observed that amounts of kinetic and potential energy of capillary waves are equal.

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